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ON A THEOREM OF KOCH

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We give a short proof of a slightly stronger version of a theorem of Koch: A complex quadratic field whose ideal class group contains a subgroup of type $(4, 4, 4)$ possesses an infinite unramified Galois pro-2 extension.

1. Koch's Theorem.

If K is a finite extension of $\mathbb{Q}$ and p is a prime number, let $K^{(0)} = K$ and for $n \geq 1$ define $K^{(n)}$ to be the maximal abelian unramified p-extension of $K^{(n-1)}$. The smallest n such that $K^{(n)} = K^{(n+1)}$ is called the length of the p-class field tower of K; if no such integer n exists, we say that K has infinite p-class field tower. By a group of type $(m_1, \dots, m_t)$ we understand a group isomorphic to $\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_t\mathbb{Z}$. The purpose of this note is to give a short proof of (a slightly strengthened version of) a theorem of Koch [**4**]:

Theorem 1. *If K is a complex quadratic field whose ideal class group contains a subgroup of type $(4, 4, 4)$, then the 2-class field tower of K is infinite.*

Koch's proof proceeds by showing that in a minimal presentation of the Galois group of the maximal unramified 2-extension of K by a free pro-2 group G, the relations lie deep in the Zassenhaus filtration of G. We replace this key ingredient of his proof, which can be thought of as the study of the quadratic unramified extensions of the genus field of K which are central over K [**3**, Satz 1], with a simple result from genus theory. Moreover, Koch's proof requires a generalization of the Vinberg/Gashütz sharpening of the Golod-Shafarevich theorem on the structure of pro-p groups [**4**, Satz 3]; for our proof, the original Vinberg/Gashütz inequality suffices (for an account of these inequalities, see, e.g., Koch's book [**5**]). Indeed, we will need only the following result (see Martinet [**8**]):

Theorem 2. *Suppose F is a totally real field of degree n, and E is a totally complex quadratic extension of F. Let t be the number of prime ideals of F which ramify in E. The 2-rank of the ideal class group of E is at least $t - 1$. If*

$$t \geq 3 + 2\sqrt{n+1},$$

then the 2-class field tower of E is infinite.

Corollary 3. *Suppose F is a totally real degree 4 extension of $\mathbb{Q}$. If two rational primes that split completely in F ramify in a complex quadratic field L, then $E = FL$ has an infinite 2-class field tower.*

Proof. With notation as in the theorem, we have $t \geq 8 \geq 3 + 2\sqrt{4+1}$. □

Proof of Theorem 1. We know that at least four primes divide the discriminant D of K. If six or more primes divide D, then an application of Theorem 2 to $K/\mathbb{Q}$ already yields the result. Assume first that exaclty four primes divide D. By the criterion of Rédei-Reichardt [**9**] on the 4-rank of the class group of K, one knows that $D = -p_1 \cdot p_2 \cdot p_3 \cdot p_4$ where p_2, p_3, p_4 are odd primes satisfying $\left(\frac{p_i}{p_j}\right) = +1$ for $i, j > 1, i \neq j$, and one of the following is satisfied:

(I) $p_1 = 4; p_j \equiv 1 \pmod 8, j = 2, 3, 4.$
(II) $p_1 = 8; p_j \equiv 1 \pmod 8, j = 2, 3, 4.$
(III) $p_1 = 8; p_2 \equiv 7 \pmod 8; p_j \equiv 1 \pmod 8, j = 3, 4.$
(IV) $p_1 \equiv 3 \pmod 4$ is an odd prime, $p_j \equiv 1 \pmod 4, j = 2, 3, 4,$
and $\left(\frac{p_1}{p_j}\right) = +1$ for $j = 2, 3, 4.$

Incidentally, Koch's theorem was originally stated for case (IV) only. Let $F = \mathbb{Q}\left(\sqrt{p_3}, \sqrt{p_4}\right)$ and $E = F\left(\sqrt{-p_1 \cdot p_2}\right)$. In all cases, (p_2) and the unique rational prime divisior of (p_1) split completely in F. Hence, by Corollary 3, E has an infinite 2-class field tower. Since E/K is an unramified 2-extension, K has an infinite 2-class field tower as well. Now suppose exactly five primes $p_1, \ldots, p_5$ divide the discriminant of K; using the Rédei-Reichardt criterion [**9**], or its equivalent form due to Rédei [**10**], it is straightforward to check that for some i, $1 \leq i \leq 5$, we have

$$p_i \equiv 1 \pmod 4, \quad \left(\frac{p_i}{p_j}\right) = 1, j \neq i.$$

Now let $F = \mathbb{Q}\left(\sqrt{p_i}\right), E = K\left(\sqrt{p_i}\right)$; E/F is a CM-extension with 8 ramified primes. By Theorem 2, E has an infinite 2-class field tower, and so does K. □

2. Further Remarks.

Koch and Venkov [**6**] have proved that a complex quadratic field whose ideal class group has a subgroup of type (p, p, p) for some odd prime p has an infinite p-class field tower. Therefore, a complex quadratic field possesses an infinite Hilbert class field tower whenever its ideal class group contains a subgroup of type (m,m,m) with $m \geq 3$. On the other hand, the field $\mathbb{Q}\left(\sqrt{-105}\right)$, whose ideal class group is of type $(2,2,2)$, has a finite class field tower, since its root discriminant is just below the Odlyzko bound (see e.g. [**8**]). I am indebted to the referee for the above remark.

Note that the proof of Koch's theorem we have given relies only on the existence of two primes that split completely in a real biquadratic field. For instance, the primes 31, 89 split completely in $\mathbb{Q}\left(\sqrt{2},\sqrt{5}\right)$, hence $\mathbb{Q}\left(\sqrt{-2\cdot 5\cdot 31\cdot 89}\right)$ has an infinite 2-class field tower; its 2-ideal class group is of type (4,2,2).

Taussky-Todd [**12**] proved that a number field with 2-ideal class group of type (2,2) has a finite 2-class field tower of length at most 2. It is natural to ask whether there are number fields with infinite 2-class field tower whose 2-class group is of type (4,2) or (2,2,2) (simplest non-cyclic 2-groups after type (2,2)). Using a minor variation on an idea first introduced by Schoof [**11**], we now show that there are complex quadratic fields with these properties. Consider, for example, $K = \mathbb{Q}\left(\sqrt{-5\cdot 7\cdot 41\cdot 61}\right)$, which has 2-ideal class group of type (2,2,2). To show that this field has infinite 2-tower, let H_0 be the Hilbert class field of $K_0 = \mathbb{Q}\left(\sqrt{5\cdot 41\cdot 61}\right)$, a real quadratic field with class number 16. Since 7 is inert in K_0, it splits into 16 prime ideals in H_0, all of which ramify in the CM extension $L = H_0\left(\sqrt{-7}\right)$. Theorem 2 shows that L, an unramified 2-extension of K, has an infinite 2-class field tower, proving the claim. In fact, for any prime q satisfying $q \equiv 7 \pmod{5\cdot 41\cdot 61}$ (there are infinitely many such primes by Dirichlet's theorem), the same argument shows that $K_q = \mathbb{Q}\left(\sqrt{-5\cdot 41\cdot 61\cdot q}\right)$ has infinite 2-class field tower; furthermore, by Rédei-Reichardt, K_q has 2-class group of type (2,2,2).

For the second example, let $K = \mathbb{Q}\left(\sqrt{-5\cdot 11\cdot 461}\right)$; this field has 2-ideal class group of type (4,2). Observe that the rational prime ideal (11) splits into 16 prime ideals in H_0, the Hilbert class field of the real quadratic field $K_0 = \mathbb{Q}\left(\sqrt{5\cdot 461}\right)$ with class number 16. Therefore, by the same argument as above, $L = H_0\left(\sqrt{-11}\right)$, and thereby K, have infinite 2-class field tower. Let H be the 2-Hilbert class field of K. Benjamin [**1**] has shown that the 2-class field tower of a complex quadratic field E with 2-class group of type (4,2) has length at most 2 if the 2-Hilbert class field of E has elementary 2-class group $\left(E = \mathbb{Q}\left(\sqrt{-5\cdot 13}\right)\text{ is an example}\right)$. Since K has infinite 2-tower, we conclude that H does not have elementary abelian 2-class group.

Finally, note that the 2-rank of the ideal class group of L is at least 15. Using Louboutin [**7**], we compute the 2-rank of the ideal class group of the biquadratic field $E = \mathbb{Q}\left(\sqrt{-11}, \sqrt{5 \cdot 461}\right)$ to be 2. The arguments of [**2**] then show that the 2-rank of the ideal class group of L is 15, 16 or 17.

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